Assignment-II

PARTIAL DIFFERENTIATION
AND
APPLICATIONS OF PARTIAL DIFFERENTIATION

1. Find the value of n, so that the equation $V = r^n\left(3\cos^2\theta - 1\right)$ satisfies the equation $\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0.$

2. If $V = r^m$ where $r^2 = x^2 + y^2 + z^2$, show that $V_{xx} + V_{yy} + V_{zz} = m\left(m+1\right)r^{m-2}$.

3. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{\left(x+y+z\right)^2}$.

4. If $x^x y^y z^z = c$, show that at $x = y = z$, $\frac{\partial^2 z}{\partial x\partial y} = -\left[y\log ey\right]^{-1}$.

5. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u = 2\cos 3u\sin u$$

6. If $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$, then evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

7. If $x = r\cos\theta, y = r\sin\theta$, show that $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$.

8. If $u = f\left(x-y, y-z, z-x\right)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

9. If $z = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$, show that $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$.

10. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ show that $\frac{\partial\left(x,y,z\right)}{\partial\left(r,\theta,\phi\right)} = r^2\sin\theta.$

11. If $x^2 + y^2 + u^2 - v^2 = 0$ and $uv + xy = 0$ prove that $\frac{\partial\left(u,v\right)}{\partial\left(x,y\right)} = \frac{x^2-y^2}{u^2+v^2}$.

12. Let $f(x,y)$ be a function defined as $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$ show that $f_x(0,0)$ and $f_y(0,0)$ exists, although $f(x,y)$ is discontinuous at $(0,0)$.

13. Expand $e^x\cos y$ in powers of x & y as far as the terms of third degree.

14. Find the approximate value of $\sqrt{(298)^2+(401)^2}$ using Taylor's series.

15. Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

16. Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 square cm.

17. Find the points on the surface $z^2=xy+1$ nearest to the origin.

18. If $u=a^3x^2+b^3y^2+c^3z^2$ where $x^{-1}+y^{-1}+z^{-1}=1$, show that the stationary value of u is given by $x=\dfrac{\sum a}{a}, y=\dfrac{\sum a}{b}, z=\dfrac{\sum a}{c}$.

19. The period T of a simple pendulum is $T=2\pi\sqrt{\dfrac{l}{g}}$. Find the maximum error in T due to possible errors up to 1% in l and 2.5% in g.

20. The temperature T at any point (x,y,z) in space is $T=400xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2+y^2+z^2=1$.